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ENUMERATION OF LARGE COMBINATORIAL STRUCTURES.(U)

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ENUMERATION OF LARGE COMBINATORIAL STRUCTURES

Final Technical Report

By

E. M. Wright

June 1981

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EUROPEAN RESEARCH OFFICE

United States Army

London, N.W.1., England

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

(U) Eleven research papers were written under the Grant, of which five are published periodicals, three accepted for publication and three recently submitted and under consideration. This year a good approximation is found to the number of non-separable sparsely edged labelled graphs, relevant in application in statistical mechanics. ~~We study~~ the enumeration of smooth labelled graphs, where we obtain an exact form for the exponential generating function, find the di-ferential equation it satisfies and a combinatorial

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) interpretation of this equation and finally study the sparsely edged case. We remark on a surprisingly close relationship between the results for the sparsely edged cases of the non-separable and the smooth graphs. We report further on the enumeration of bipartite graphs, labelled and unlabelled, discussed in detail in the Second Annual Report, and finally list a few further problems under investigation.

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Abstract

We list eleven research papers written under the Grant, of which five have been published in scientific periodicals, three have been accepted for publication and three have been submitted and under consideration. This year I report that we find a good approximation to the number of non-separable sparsely-edged labelled graphs, relevant to applications in statistical mechanics. I also study the enumeration of smooth labelled graphs, where I obtain an exact form for the exponential generating function, find the differential equation it satisfies and a combinatorial interpretation of this equation and finally study the sparsely edged case. I remark on a surprisingly close relationship between the results for the sparsely edged case of the non-separable and the smooth graphs. I report a little further on the enumeration of bipartite graphs, labelled and unlabelled, and finally list a few of the further problems that I hope to investigate.

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Papers published, accepted or submitted

Papers published

A1. (with T.R.S.Walsh) The k-connectedness of unlabelled graphs, J.London Math. Soc. (2) 18 (1978), 397-402; Appendix 1 of First Annual Report.

A2. The Tarry-Escott and the "easier" Waring problems, J.fuer reine u. angew. Math. 311/312 (1979), 170-173; Appendix 6 of first Annual Report.

A3. A quadratic sequence of Faltung type, Math. Proc, Cambridge Phil. Soc. 88(1980), 193-197; Appendix 1 of Second Annual Report.

A4. The number of connected sparsely-edged graphs III Asymptotic results, J.Graph Theory 4(1980),393-407; Appendix 3 of First Annual Report.

A5. Burnside's Lemma: an historical note, J. Combin. Theory. (B) 30 (1981),89-90; Appendix 5 of First Annual Report.

Papers accepted for publication

A6. The k-connectedness of bipartite graphs, J. London Math. Soc; Appendix 3 of this Report.

A7. (with V.L.Klee and D.G.Larman) The proportion of labelled bipartite graphs which are connected, J. London Math. Soc.; Appendix 3 of Second Annual Report.

A8. (with V.L.Klee and D.G.Larman) The diameter of almost all bipartite graphs, Stud. Sci. Math. Hungarica (to appear in 15, Nos. 1-2); in draft form as Appendix 4 of Second Annual Report.

Papers submitted for publication

A9. The number of connected sparsely edged graphs IV Large non-separable graphs, J. Graph Theory; Appendix 1 of this Report.

A10. Enumeration of smooth labelled graphs, Proc. Roy. Soc. Edinburgh A (Mathematics); Appendix 2 of this Report.

A11. The number of sparsely-edged Hamiltonian graphs, Glasgow Math. J.; Appendix 4 of First Annual Report.

Postgraduate students.

There were no postgraduate students working under this Grant. Paper A1 above was written in collaboration with T.R.S.Walsh, who is a postgraduate student at the computer Centre of the USSR Academy of Sciences in Moscow.

Note on references

References [A1], [A2],etc. are to papers in the preceding list. References [B1], [B2] ,etc. are to the list of references at the end of the Report (pp.67,68).

<u>The number of large non-separable labelled graphs</u>

1. An (n,q) graph is a graph on n labelled points and q lines. A <u>non-separable</u> graph (or <u>block</u>) is a connected graph which cannot be disconnected by the removal of one point and its adjacent lines. We write $u(n,q)$ for the number of non-separable graphs on n labelled points and q lines and are concerned to find an asymptotic approximation to $u(n,n+k)$ as $n \to \infty$ and $k \to \infty$ with $k = o(n^{\frac{1}{2}})$. The determination of such an approximation is related to problems in statistical mechanics (see [B1], [B6], esp. p.141, and [B7]).

We write

$$U_k = \sum_n u(n,n+k)X^n/n!$$

for the exponential generating function (e.g.f.) of $u(n,n+k)$. In [B13] I found a recurrence differential equation which enabled one, in theory, to determine U_k for successive k. The work could be carried out on a computer and would give one the coefficients in the expression

$$U_k = b_k \phi^{-3k} - c_k \phi^{-3k+1} + \sum_{s=-3k+2}^{2} c_{ks} \phi^{s},$$

where $\phi = 1-X$. Hence one could find an expression for $u(n,n+k)/n!$ in terms of binomial coefficients. But this expression contains $3k+3$ terms and the exact evaluation of the coefficients rapidly e hausts the memory of the computer, so this procedure is not practicable for any but small k, nor would the result be very informative for such k. However I can use the recurrenec differential equation first to prove that

$$\left[b_k \phi^{-3k} - c_k \phi^{-3k+1}\right]_n \leq \left[U_k\right]_n \leq \left[b_k \phi^{-3k}\right]_n, \qquad (1.1)$$

where $[F]_n$ denotes the coefficient of $X^n/n!$ in the power series F, and secondly to find recurrence formulae for the sequences $\{b_k\}$ and $\{c_k\}$. We can use these to show that $c_k = (3k+1)kb_k/(3k-1)$ and so that, for $k \geq 2$,

$$b_k(n+3k-2)!(n-3k^2+2k-1) \leq (3k-1)!u(n,n+k)$$
$$\leq b_k(n+3k-1)! \qquad (1.2)$$

We can transform the recurrence satisfied by the sequence $\{b_k\}$ into a particular case of the quadratic recurrence studied by Stein and Everett [B5] and solved by me [A3]. My solution leads to the result that

$$b_k \sim a_1(3/2)^k(k-1)! \qquad (1.3)$$

as $k \to \infty$, where $a_1 = 0.058538\ldots$ (the value of a_1 obtained by computing, using a theorem of [A3] to make the work manageable). We have then

$$u(n,n+k) \sim a_1(3/2)^k(k-1)!(n+3k-1)!/(3k-1)! \qquad (1.4)$$

as $n,k \to \infty$ with $k = o(n^{\frac{1}{2}})$. From this, using Stirling's formula, we can deduce that

$$u(n,n+k) \sim a_1(6\pi)^{\frac{1}{2}}n^{n+3k-\frac{1}{2}}e^{2k-n}(18k^2)^{-k}. \qquad (1.5)$$

The inequalities (1.2), very precise for $k = o(n^{\frac{1}{2}})$, give some ground for optimism over their use in the applications mentioned above.

We can go a little further and show that, if $k < (1-\varepsilon)(\tfrac{1}{3}n)^{\frac{1}{2}}$, where ε is a positive number independent of k and n, then $\log u(n,n+k)$ has the asymptotic approximation corresponding to (1.5) with error $O(1)$.

The details of this work are embodied in Appendix 1 of this report, which is a paper [A9] which has been submitted for possible publication in the Journal of Graph Theory.

Enumeration of smooth labelled graphs

2. This section is an expansion of §5 of the Second Annual Report, containing further results. The full details are in Appendix 2, which is a paper [A10] submitted for possible publication to the Proceedings of the Royal Society of Edinburgh.

A smooth graph is a connected graph without end points. Let $v(n,q)$ be the number of smooth labelled (n,q) graphs. The e.g.f. of $v(n,q)$ is

$$V(Z,Y) = \sum_{n\geq 1} v(n,q) Z^n Y^q / n!$$

By the core and mantle method due to Riddell and to Ford and Uhlenbeck (see [B3]), I find the functional equation satisfied by V. In this case, unlike other applications of the method, the inverse of the auxiliary e.g.f. introduced can be expressed in simple terms and so I find an explicit form for V, namely

$$V(Z,Y) = \log\left\{1 + \sum_{n\geq 1} Z^n e^{-nZY}(1+Y)^{\frac{1}{2}n(n-1)}\right\} - Z + \tfrac{1}{2}Z^2 Y.$$

From this I can find a partial differential equation (p.d.e.) satisfied by V, namely

$$\begin{aligned} 2(1-ZY)^3(1+Y)V_Y &= Z^2(1-ZY)(V_{ZZ}+V_Z^2) \\ &\quad + Z^3Y^2(3-2ZY)V_Z + Z^3Y^2(1-ZY)^2. \end{aligned}$$

I can also find a direct combinatorial proof (or, one might say, a combinatorial interpretation) of the p.d.e. I give this in full in Appendix 2. Again, if we write

$$V = \sum_{k=0}^{\infty} V_k Y^k,$$

where

$$V_k = \sum_{n=1}^{\infty} v(n,n+k)(ZY)^n/n!,$$

we can find the differential recurrence formula for V_k as k increases.

A number of asymptotic results follow for $v(n,q)$. We write

$$\mu \equiv (q/n) - \tfrac{1}{2}\log n - \tfrac{1}{2}\log\log n.$$

If $\mu \to c$ as $n \to \infty$, we can deduce trivially from the work of Erdös and Rényi [B2] that the proportion of labelled (n,q) graphs which are smooth tends to $\exp(-e^{-2c})$ as $n \to \infty$. If $\mu \to +\infty$, we can apply the method of [B8] to the explicit form of $V(Z,Y)$ above so as to find an asymptotic <u>expansion</u> for $v(n,q)$ and so also for the relatively small number of (n,q) graphs which are not smooth for these q. See Theorem 4 of Appendix 2. This result can also be found, but with rather more difficulty, by the use of the Inclusion-Exclusion Theorem.

Finally I can apply the method of [A4] (almost word for word, if I replace θ b 1-X) to the recurrence formula for V_k to find an asymptotic approximation to $v(n,n+k)$ if $k = o(n^{\frac{1}{2}})$ as $n \to \infty$, viz.

$$v(n,n+k) \sim b'_k(n+3k-1)!/(3k-1)!$$

(In Appendix 2, we write b_k, but, to avoid confusion with Appendix 1, I use b'_k here). We find that

$$b'_k \sim d(3/2)^k(k-1)! \qquad (2.1)$$

as $k \to \infty$, where $d = 0.159155...$ Hence

$$v(n,n+k) \sim d(3/2)^k (k-1)!(n+3k-1)!/(3k-1)! \tag{2.2}$$

and

$$v(n,n+k) \sim d(6\pi)^{\frac{1}{2}} n^{n+3k-\frac{1}{2}} e^{2k-n} (18k^2)^{-k} \tag{2.3}$$

as $k, n \to \infty$ with $k = o(n^{\frac{1}{2}})$.

A comparison of these results

3. It follows easily from Erdős and Rényi's work in [B2] that, if $\mu \to +\infty$ as $n \to \infty$, then

$$v(n,q) \sim u(n,q) \sim N!/\, q!(N-q)!\ ,$$

where $N = \frac{1}{2}n(n-1)$, i.e. almost all labelled (n,q) graphs are non-separable and smooth. It is easy to see from [B2] how this happens. If, on the contrary, we consider $v(n,n+k)$ and $u(n,n+k)$ when $k \to \infty$ with n but $k = o(n^{1/3})$, we have, from (1.4) and (2.2),

$$v(n,n+k)/u(n,n+k) \sim b'_k/b_k \to d/a_1 = 2.71883\ldots \qquad (3.1)$$

This is puzzling. The sequences $\{b'_k\}$ and $\{b_k\}$ are defined by the recurrences

$$2(k+1)b'_{k+1} = 3k(k+1)b'_k + 3B'_k, \qquad (3.2)$$

where $b'_1 = 5/24$, $b'_2 = 5/16$ and

$$B'_k = \sum_{s=1}^{k-1} s(k-s)b_s b_{k-s}$$

and by

$$2(k+1)b_{k+1} = (3k+2)(kb_k+3B_k), \qquad (3.3)$$

where $b_1 = 1/12, B_2 = 5/48$ and

$$B_k = \sum_{s=1}^{k-1} s(k-s)b_s b_{k-s}.$$

These recurrences look similar and they have the very similar asymptotic solutions (1.3) and (2.1). But it does not seem possible to transform one into the other nor even to obtain any relation between them, The two papers [A9] and [A10] were drafted at different times of the year and the two recurrences (3.2) and (3.3) were solved asymptotically in very different ways (see [A3] and [A7]). Thus the close similarity between (1.4) and (2.2) did not strike me until I started collecting material for this report. A further point which only struck me while I was actually writing the report is that the number 2.71883... in (3.1) is very nearly equal to e = 2.71828..., the base of Napierian logarithms (in fact, it only differs by 2 parts in 10000). The obvious conjecture is that there has been a minor error in calculating d or a_1 and that the ratio should be exactly e.

I have made no reference to all this in [A9] and [A10] (Appendices 1 and 2) since, at present, I do not understand it. It seems very unlikely that it is no more than a coincidence that v(n,n+k) and u(n,n+k) have such similar asymptotic approximations and these in a constant ratio (probably e) to one another. Clearly we must recalculate d and a_1 (though we had certainly checked the calculations very carefully). But the relationship between v(n,n+k) and u(n,n+k) for this range of k is more intriguing.

Bipartite graphs

4. The referee for my paper [A6] first reported that "most of the results were contained in stronger ones proved by Bollobás in a paper to be published in the Canadian J. Math". Dr. Bollobás was kind enough to let me have a preprint of his paper, when it turned out that the overlap between his results and mine was very small (and in this small overlap, amusingly enough, my proof was under wider conditions). He had a beautiful theorem which I had not, namely, an asymptotic formula for the connectivity of almost all bipartite graphs on m labelled red and n labelled blue points when m/n is bounded above and from zero below as $m,n \rightarrow \infty$; in fact, his result also covers multipartites. On the other hand, the main interest of my paper was in unlabelled graphs and included the case when $m/n \rightarrow 0$, neither of which did he touch. (The referee was presumably relying on his memory of Bollobás's paper.) When I pointed out these differences, the referee recommended my paper for publication and it has been accepted. A revised form which refers to Bollobás and includes a further minor result appears as Appendix 3 of this report.

The further result is that, if $\alpha = \alpha(m,n)$ is the proportion of (m,n) bipartites which are connected, then (i) if $n2^{-m} \rightarrow w$ as $m,n \rightarrow \infty$ with $m < n$, then $\alpha \rightarrow e^{-w}$ and (ii) if $n2^{-m} \rightarrow \infty$, then $\alpha \rightarrow 0$. Dr. Bollobás says that he knows this result and that he

thinks he has seen a published proof by somebody.
A conjecture which suggests itself to me and which seems likely, but which I cannot yet prove, is as follows. If $m < n$, then α increases with m and decreases as n increases.

Professor Erdős informed me that I. Palasti had written on connectedness in bipartites. Her paper [B4] proves the following (in my notation). Let $m/n \to \lambda$ as $n \to \infty$, where $0 < \lambda \leq 1$ and let $E = [n \log n + cn]$. Then the proportion of labelled $(m,n;E)$ bipartites which are connected tends to $\exp(-ge^{-c})$, where $g = 1$ if $\lambda \neq 1$ and $g = 2$ if $\lambda = 1$. This can be deduced quite simply as a special case of our result from [A7] stated in Theorem 6 of Appendix 3. Our theorem also makes the apparent "discontinuity" of g when $m = n$ less surprising.

To me the most interesting (and the most difficult) problems in the enumeration of bipartite graphs remain those which occur in the unlabelled case and especially when $m/n \to 0$ as $m,n \to \infty$. I have worked on these this year and think that I begin to see daylight, but my results are as yet too fragmentary to be worth reporting. They give some indication already, however, that phenomena may occur as interesting and as surprising as in the case of ordinary random graphs (see, for example, [B9,B10,B11,B12,A1]).

Some further problems

5.1. I am actively engaged in investigating the problems mentioned at the ends of §3 and §4.

5.2. I still have to write up for publication (if possible in a simplified form) the proof of my results on the behaviour of $\beta(n,q)$, the proportion of unlabelled graphs on n points and q lines which are connected, as q increases. These results were announced in [B9,B10]; both the nature of the results and the fact that so much can be found are surprising. But the methods are elaborate and the details complicated.

5.3. Dr. Sheehan and I have not yet found time to apply further our use of the idea of a "ghost" asymptotic expansion nor indeed to publish an account of the method.

5.4. There remain the possible applications of the result of §1 to a problem in Statistical mechanics.

Appendix 1

The number of connected sparsely-edged graphs IV
Large non-separable graphs.

E.M.Wright
University of Aberdeen

(Submitted to J. Graph Theory)

Abstract

The number of non-separable graphs on n labelled points and q lines is $u(n,q)$. In the second paper of this series I showed how to find an exact formula for $u(n,n+k)$ for general n and successive (small) k. The method would give an asymptotic approximation for fixed k as $n \to \infty$. Here I find an asymptotic approximation to $u(n,n+k)$ when $k = o(n^{1/2})$ and an approximation to $\log u(n,n+k)$ when $k < (1-\varepsilon)\sqrt{(n/3)}$. The problem of finding an approximation to $u(n,q)$ when $(q-n)/n^{1/2} \to +\infty$ and $(q/n) - \frac{1}{2}\log n - \frac{1}{2}\log\log n \to -\infty$ is open.

Introduction

1. An (n,q) graph is a simple graph on n labelled points and q lines (no loops, no multiple lines). Such a graph is said to be <u>non-separable</u> (or a <u>block</u> or <u>2-connected</u>) if it is connected and cannot be disconnected by the removal of any one point and its adjacent lines. We write $u(n,q)$ for the number of non-separable (n,q) graphs. The determination of $u(n,q)$ and, in particular, of an asymptotic approximation for large, almost equal n and q is related to problems in statistical mechanics (see [1], [4, esp. p.141] and [5]).

We put $\mu = (q/n) - \frac{1}{2}\log n - \frac{1}{2}\log\log n$. Erdős and Rényi [2] proved that, if $\mu \to c$ as $n \to \infty$, where c is a fixed number, then the proportion of (n,q) graphs which are non-separable tends to $\exp(-e^{-2c})$. It can readily be shown that, for fixed n, the proportion increases (at least in the non-strict sense) with q. Hence, if $\mu \to +\infty$, almost all (n,q) graphs are 2-connected, while, if $\mu \to -\infty$, almost none are 2-connected.

It is trivial that $u(n,q) = 0$ if $q < n-1$. Again $u(2,1) = 1$, but $u(n,n-1) = 0$ if $n > 2$, and $u(n,n) = \frac{1}{2}\{(n-1)!\}$ if $n \geq 3$. In [7] I found a method to calculate an exact formula for $u(n,n+k)$ for successive $k \geq 1$; for example

$$24u(n,n+1) = (n-3)(n+2)n! \qquad (n \geq 3)$$

The method can be carried out by a computer (see [3,6] for a similar method), but as the resulting formula has $3k+3$ terms, it is not very informative for substantial k. For bounded k it does yield an asymptotic formula as $n \to \infty$. Here however I develop the method further so as to obtain the following theorem.

__Theorem 1.__ __For all__ $k \geq 2$,

$$b_k(n+3k-2)!(n-3k^2+2k-1) \leq (3k-1)!\, u(n,n+k)$$

$$\leq b_k(n+3k-1)!, \qquad (1.1)$$

__where__ $\qquad b_1 = 1/12, \quad b_2 = 5/48 \qquad (1.2)$

$$B_k = \sum_{s=1}^{k-1} s(k-s)b_s b_{k-s} \qquad (k \geq 2) \qquad (1.3)$$

__and__

$$2(k+1)b_{k+1} = (3k+2)(kb_k + 3B_k) \qquad (k \geq 2). \quad (1.4)$$

If $3k^2-2k+1 \geq n$, the left-hand inequality in (1.1) tells us nothing new, since obviously $u(n,n+k) \geq 0$. For smaller k, however, we can immediately deduce the following theorem.

__Theorem 2.__ __If__ $2 \leq k = o(n^{1/2})$, __then__

$$u(n,n+k) = b_k\{(n+3k-1)!/(3k-1)!\}\{1 + O(k^2/n)\},$$

__as__ $n \to \infty$.

Again, if $2 \leq k < (1-\varepsilon)\sqrt{(n/3)}$, where ε is a positive number independent of n, then

$$\log u(n,n+k) = \log\{b_k(n+3k-1)!/(3k-1)!\} + O(1). \qquad (1.5)$$

If $k \to \infty$ as $n \to \infty$, Theorem 2 and (1.5) are again uninformative unless we can "solve" (1.4) asymptotically. The solution is as follows.

<u>Theorem 3</u>. <u>As</u> $k \to \infty$,

$$b_k = a_1(3/2)^k(k-1)!\{1 + O(k^{-1})\}, \qquad (1.6)$$

<u>where</u> $a_1 = 0.058538\ldots$

Using the well-known approximation to the factorial of a large number (Lemma 1 of [8], since $t! = \Gamma(t+1)$), we obtain the following two results from Theorems 2 and 3 and (1.5).

<u>Theorem 4. If</u> $k \to \infty$ <u>as</u> $n \to \infty$ <u>but</u> $k = o(n^{1/2})$, <u>then</u>

$$u(n,n+k) = a_2 n^{n+3k-\frac{1}{2}} e^{2k-n}(18k^2)^{-k}\{1 + O(k^{-1}) + O(k^2/n)\},$$

<u>where</u> $a_2 = a_1\sqrt{(6\pi)} = 0.25415\ldots$

<u>Theorem</u> 5. <u>If</u> $k \to \infty$ <u>as</u> $n \to \infty$, <u>but</u> $k < (1-\varepsilon)\sqrt{(n/3)}$, <u>where</u> ε <u>is a positive number independent of</u> n, <u>then</u>

$$\log u(n,n+k) = (n+3k-\tfrac{1}{2})\log n - k \log(18k^2) - n + 2k + O(1).$$

As we indicate in §4, we can obtain a closer approximation to b_k than in (1.6) and so improve the error term $O(k^{-1})$ in Theorem 4 at the cost of a little complication. There remains the problem of finding an asymptotic approximation to $u(n,n+k)$ when $k/n^{1/2} \to \infty$ and $\mu \to -\infty$. This seems difficult and I have no ideas towards a solution.

We have now to prove Theorems 1 and 5.

2. Fundamental lemmas

We write

$$U_k = U_k(X) = \sum_n u(n,n+k)X^n/n!,$$

the exponential gen rating function of u(n,n+k). (The power series converges when $|X| < 1$, but we do not need this and in fact treat U_k as a formal series.) Dashes denote differentiation with respect to X. We write $\phi = 1-X$ and, for shortness, $\eta_k = XU'_k + kU_k$.

Lemma 1. For all $k \geq 2$, we have

$$2\eta_{k+1} = J_k, \qquad (2.1)$$

where

$$J_k = \phi^{-2}X^2\{U''_k + X^2U''_{k-1} + (\phi^{-1}+1)(\eta_k+\eta_{k-1}) -2\eta_k$$
$$+2X\phi^{-1}(T_k + T_{k-1}) \qquad (2.2)$$

and $\quad T_k = \sum_{s=1}^{k-1} U''_s\eta_{k-s} \quad (k \geq 2), \quad T_1 = 0.$

Lemma 1 is immediate if, in [7], we substitute from (4) in (5), equate coefficients of Y^{k+1} and divide through by ϕ. Lemma 1 can be proved by a direct combinatorial argument, but this is inevitably longer.

In [7] we showed that U_k can be expressed as a finite sum of powers of ϕ, mainly negative. For example,

$$12U_1 = \phi^{-3} - 2\phi^{-2} - 2\phi^{-1} + 8 - 7\phi + 2\phi^2 \qquad (2.3)$$

and

$$48U_2 = 5\phi^{-6} - 14\phi^{-5} + 7\phi^{-4} + 8\phi^{-3} - 3\phi^{-2} + 2\phi^{-1} - 19 + 20\phi - 6\phi^2. \quad (2.4)$$

For $k \geq 1$, we have

$$U_k = b_k \phi^{-3k} - c_k \phi^{-3k+1} + \sum_{t=-2}^{3k-2} u_{kt} \phi^{-t}, \qquad (2.5)$$

where we now take (2.5) as the <u>definition</u> of the sequences $\{b_k\}$ and $\{c_k\}$ and have to prove (1.2) and (1.4).

<u>Lemma 2. The sequence $\{b_k\}$ defined by</u> (2.5) <u>satisfies</u> (1.2) <u>and</u> (1.4). <u>Also</u>

$$c_k = (3k+1)kb_k/(3k-1). \qquad (2.6)$$

From (2.3) and (2.4) we see that (1.2) is true and that (2.6) is true for $k = 1$ and $k = 2$. We substitute from (2.5) in (2.1) and equate the coefficients of ϕ^{-3k-4}. After trivial calculations, in which we use the fact that, for any sequence $\{\alpha_t\}$, we have

$$\sum_{t=1}^{k-1} t\alpha_t \alpha_{k-t} = \sum_{t=1}^{k-1} (k-t)\alpha_t \alpha_{k-t} = \frac{1}{2} k \sum_{t=1}^{k-1} \alpha_t \alpha_{k-t}, \qquad (2.7)$$

we obtain (1.4). Again, if we equate the coefficients of ϕ^{-3k-3} and use (1.4) and (2.7), we obtain

$$2(3k+2)c_{k+1} = 6(k+1)b_{k+1} + (3k+1)kb_k + (3k-1)(3k+1)c_k$$
$$+ 6(3k+1)\sum_{s=1}^{k-1} (k-s)(3s-1)b_{k-s}c_s. \qquad (2.8)$$

Now (1.2), (1.4) and (2.8) together fix the value of c_k for all $k \geq 1$. We know that (2.6) is true for $k = 1$ and $k = 2$. If we substitute from (2.6) in (2.8), the latter reduces to (1.4). Hence (2.6) is true for all $k \geq 1$. ∎

If

$$F = F(X) = \sum_{n \geq 0} f(n) X^n/n!,$$

we write $[F]_n = f(n)$. If $[F]_n \geq 0$ for all $n \geq 0$, we say that $F \geq 0$. Similarly $F_1 \geq F_2$ means that $[F_1]_n \geq [F_2]_n$ for all $n \geq 0$. Since we treat all power series in X as formal power series and never consider the value of F for a particular value of X, no confusion arises from the notation. Theorem 1 follows from Lemma 2 and the following lemma, which we have still to prove.

<u>Lemma 3</u>. <u>We have</u>

$$b_1 \phi^{-3} - c_1(\phi^{-2} + \phi^{-1}) \leq U_1 \leq b_1 \phi^{-3} \qquad (2.9)$$

<u>and</u>

$$b_k \phi^{-3k} - c_k \phi^{-3k+1} \leq U_k \leq b_k \phi^{-3k} \qquad (k \geq 2). \qquad (2.10)$$

3. Proof of Lemma 3.

(2.9) follows trivially from (2.3). To deduce (2.10) for $k = 2$ from (2.4), it is enough to show that

$$7\phi^{-4}+8\phi^{-3}-3\phi^{-2}+2\phi^{-1}-19+20\phi-6\phi^{2}\geq 0$$

and

$$14\phi^{-5}-7\phi^{-4}-8\phi^{-3}+3\phi^{-2}-2\phi^{-1}+19-20\phi+6\phi^{2}\geq 0,$$

which are also trivial.

To prove Lemma 3 by induction on k, we have now only to prove the following lemma.

Lemma 4. *If* (2.10) *is true for* $2\leq k\leq j$, *where* $j\geq 2$, *then* (2.10) *is true for* $k = j+1$.

We write $D = d/dX$ and remark first that the operator $XD + k$, applied to any formal power series, multiplies the coefficient of X^n by $n+k$. If $k\geq 1$, it follows that $XD + k$ is a bipositive operator, i.e.

$$F\geq 0 \Leftrightarrow (XD+k)F\geq 0$$

for any formal power series F. Similarly

$$F_1\geq F_2 \quad \Leftrightarrow \quad (XD+k)F_1\geq (XD+k)F_2. \tag{3.1}$$

Now

$$(XD+k)\phi^{-3k} = k\phi^{-3k-1}(1+2X) \tag{3.2}$$

and so, by (3.1), to prove Lemma 4, it is enough to show that

$$2R_{j+1} \leq J_j \leq 2(j+1)b_{j+1}\phi^{-3j-4}(1+2X) \qquad (3.3)$$

follows from the hypothesis of Lemma 4, where

$$R_k = (kD+k)(b_k - c_k\phi)\phi^{-3k}$$
$$= 3kb_k\phi^{-3k-1}\{1-(k+1)\phi\} + (2k-1)c_k\phi^{-3k+1}. \qquad (3.4)$$

We now assume the hypothesis. For $1 \leq k \leq j$, we have
$U_k \leq b_k\phi^{-3k}$, $U_k' \leq 3kb_k\phi^{-3k-1}$, $U_k'' \leq 3k(3k+1)b_k\phi^{-3k-2}$
and, by (3.1) and (3.2),

$$\eta_k \leq kb_k\phi^{-3k-1}(1+2X) \leq 3kb_k\phi^{-3k-1}. \qquad (3.5)$$

Hence

$$2T_k \leq 6(1+2X)\phi^{-3k-3}\sum_{s=1}^{k-1} s(k-s)(3s+1)b_s b_{k-s}$$
$$= 3(3k+2)B_k(1+2X)\phi^{-3k-3}$$

by (2.7).

We have then $J_j \leq E_j$, where

$$E_j = X^2\phi^{-3j-4}\{3(3j+1)jb_j + 3(3j-2)X^2(j-1)b_{j-1}\phi^3$$
$$+(1+\phi)(1+2X)(jb_j + (j-1)b_{j-1}\phi^3)\}$$
$$+ 3X\phi^{-3j-4}(1+2X)\{(3j+2)B_j + (3j-1)B_{j-1}\phi^3\}$$

and $B_1 = 0$. To prove the right-hand inequality in (3.3), it is enough to show that

$$E_j \leq 2(j+1)b_{j+1}\phi^{-3j-4}(1+2X). \qquad (3.6)$$

But, if we use (1.3) and rearrange, we see that (3.6) is equivalent to

$$K_1(j+1)b_{j+1} + K_2 j b_j + K_3(j-1)b_{j-1} \geq 0,$$

where

$$K_1 = 2(1+2X)\phi^{-3j-3} \geq 0,$$
$$K_2 = X(3j+X)\phi^{-3j-3} + 4X^3\phi^{-3j-2} \geq 0,$$
$$K_3 = (3j-2)X\phi^{-3j}(1+3X+3x^2) + X\phi^{-3j+1}(1+2X) \geq 0,$$

and the inequality follows.

To prove the left-hand inequality in (3.3), we require the following minor lemma.

<u>Lemma 5</u>. <u>If</u> F_i $(1 \leq i \leq 6)$ <u>are power series such that</u> $F_i \geq 0$ <u>and</u>

$$F_1 \geq F_2 - F_3, \qquad F_4 \geq F_5 - F_6,$$

<u>then</u>

$$F_1F_4 \geq F_2F_5 - F_3F_5 - F_2F_6. \qquad (3.7)$$

If $x_1, \ldots, x_6$ are non-negative numbers and if $x_1 \geq x_2 - x_3$ and $x_4 \geq x_5 - x_6$, then

$$x_1x_4 \geq x_2x_5 - x_3x_5 - x_2x_6. \qquad (3.8)$$

We may now put

$$x_i = [F_i]_s \quad (1 \leq i \leq 3), \quad x_i = [F_i]_{n-s} \quad (4 \leq i \leq 6)$$

by the hypothesis of Lemma 5. Summing each side of (3.8) from $s = 0$ to $s = n$, we find that

$$[F_1F_4]_n \geq [F_2F_5 - F_3F_5 - F_2F_6]_n$$

for all $n \geq 0$, that is (3.7). ∎

By the hypothesis of Lemma 4, for $2 \leq k \leq j$, we have

$$U_k \geq b_k \phi^{-3k} - c_k \phi^{-3k+1}, \qquad U_k' \geq kb_k\{3\phi^{-3k-1} - (3k+1)\phi^{-3k}\},$$

$$U_k'' \geq 3k(3k+1)b_k(\phi^{-3k-2} - k\phi^{-3k-1}), \tag{3.9}$$

$$\begin{aligned} \eta_k = xU_k' + kU_k &\geq 3kb_k\{\phi^{-3k-1} - (k+1)\phi^{-3k}\} + k\phi^{-3k+1}\{(3k+1)b_k - c_k\} \\ &\geq 3kb_k\{\phi^{-3k-1} - (k+1)\phi^{-3k}\}, \end{aligned} \tag{3.10}$$

since

$$c_k = k(3k+1)b_k/(3k-1) < (3k+1)b_k$$

by (2.6). Again

$$U_1 \geq b_1(\phi^{-3} - 2\phi^{-2} - 2\phi^{-1}), \qquad \eta_1 \geq 3b_1(\phi^{-4} - 2\phi^{-3}),$$

which is (3.10) for $k = 1$, and

$$U_1'' \geq 4b_1(3x\phi^{-5} - \phi^{-3}).$$

Hence, if $2 \leq k \leq j-1$, by Lemma 5,

$$U_k''\eta_{j-k} \geq 9(3k+1)k(j-k)b_k b_{j-k}\{\phi^{-3j-3} - (j+1)\phi^{-3j-2}\}. \tag{3.11}$$

Again, by Lemma 5,

$$\begin{aligned} U_1''\eta_{j-1} &\geq 12(j-1)b_1 b_{j-1}\phi^{-3j-3}(3x - 3jx\phi - \phi^2) \\ &\geq 36(j-1)b_1 b_{j-1}\{\phi^{-3j-3} - (j+1)\phi^{-3j-2}\}, \end{aligned}$$

which is (3.11) with $k = 1$. Hence, by (3.11) and (2.7),

$$2T_j \geq 9(3j+2)B_j\phi^{-3j-3}\{1 - (j+1)\phi\}. \tag{3.12}$$

Clearly, by (2.2),

$$J_j \geq x^2U_j''\phi^{-2} + x^2(\phi^{-3} + \phi^{-2})\eta_j + 2x\phi^{-1}T_j - 2\eta_j$$

and so, by (3.9), (3.10), (3.12) and (3.5),

$$J_j \geq xK_4 - 2jb_j\phi^{-3j-1},$$

where

$$K_4\phi^{3j+4} = 3j(3j+1)b_j(1-j\phi)(1-\phi) + 3j\,b_j\{1-(j+1)\phi\}(1-\phi^3)$$
$$+\,9(3j+2)B_j\{1-(j+1)\phi\} - 4j\,b_j\,\phi^3.$$

Using (1.4) and simplifying, we find that

$XK_4 = 6(j+1)b_{j+1}K_5 + jb_jXK_6 \geq 6(j+1)b_{j+1}K_5$

where $K_5 = \{1-(j+1)\phi\}(1-\phi)\phi^{-3j-4}$ and

$K_6 = 3(3j^2+j-1)\phi^{-3j-2} + (3j-1)\phi^{-3j-1} \geq 0.$

To prove that $J_j \geq 2R_{j+1}$, it is then enough by (3.4) to show that

$$6(j+1)b_{j+1}K_5 - 2j\,b_j\,\phi^{-3j-1}$$
$$\geq 6(j+1)b_{j+1}\phi^{-3j-4}\{1-(j+2)\phi\} + 2(2j+1)c_{j+1}\phi^{-3j-2},$$

that is

$$\{3(j+1)^2b_{j+1} - (2j+1)c_{j+1}\}\phi^{-3j-2} \geq j\,b_j\,\phi^{-3j-1}. \qquad (3.13)$$

Now, by (2.6) and (1.4),

$3(j+1)^2b_{j+1} - (2j+1)c_{j+1} = (j+1)b_{j+1}(3j^2+4j+2)/(3j+2)$

But $\phi^{-3j-2} \geq \phi^{-3j-1} \geq 0$ and (3.13) follows. ∎

4. Proof of Theorem 3: the behaviour of b_k.

If we write $S_1 = 1$, $S_2 = 2/3$ and

$$S_{k+1} = 2^{2k+1}3^{1-k}kb_k \qquad (k \geq 1),$$

we find that (1.2) and (1.4) are equivalent to

$$S_{k+1} = (k-\tfrac{1}{3})\sum_{s=1}^{k} S_s S_{k+1-s} \qquad (k \geq 1), \qquad S_1 = 1.$$

This is the particular case ($b = -\frac{1}{3}$) of the more general recurrence formula studied in [9]. Theorem 2 of [9] gives us Theorem 3 of the present paper, computation supplying the value of a_1. We can in fact deduce that

$$b_k = a_1\left(\frac{3}{2}\right)^k (k-1)!\left\{1 - \frac{7}{9k} - \frac{132}{162k(k-1)} - O\left(\frac{1}{k^3}\right)\right\}$$

or

$$b_k = a_1\left(\frac{3}{2}\right)^k\left\{(k-1)! - \frac{7}{9}(k-2)! - \frac{11}{162}(k-3)! - O\left((k-4)!\right)\right\}$$

for large k (or indeed find further terms of the asymptotic expansion). ∎

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Appendix 2

Enumeration of smooth labelled graphs

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Synopsis

An (n,q) graph is a graph on n labelled points and q lines without loops or multiple lines. We write v(n,q) for the number of smooth (n,q) graphs, i.e. connected graphs without end points, and $V = V(Z,Y) = \sum_{n,q} v(n,q)Z^nY^q/n!$ for the exponential generating function of v(n,q). We use the Riddell "core and mantle" method to find an explicit form for V (not, as usual with this method, only a functional equation). From this we deduce a partial differential equation satisfied by V. We interpret this equation in purely combinatorial terms. We write $V_k = \sum_n v(n,n+k)X^n/n!$ and find a recurrence formula for V_k for successive k. We use these and other results to find an asymptotic expansion for v(n,q) as $n \to \infty$ when $(q/n) - \frac{1}{2}\log n - \frac{1}{2}\log\log n \to +\infty$ and an

asymptotic approximation to $v(n,n+k)$ when $0 < k = o(n^{1/2})$ and to $\log v(n,n+k)$ when $k < (n/3)^{1/2}(1-\varepsilon)$.

1. Introduction

An (n,q) graph is a graph on n labelled points and q lines without loops or multiple lines. A <u>smooth</u> graph is a connected graph on 3 or more points without end points. We write $N = \frac{1}{2}n(n-1)$ and $B(h,k) = h!/\{k!(h-k)!\}$, so that $B(N,q)$ is the number of (n,q) graphs. Hence the exponential generating function (e.g.f.) of this number is

$$R = R(Z,Y) = 1 + \sum_{n=1}^{\infty} Z^n(1+Y)^N/n!$$

Again $f(n,q)$ is the number of connected (n,q) graphs and $v(n,q)$ the number of these which are smooth; the respective e.g.f.s are

$$F = F(Z,Y) = \sum_{n=1}^{\infty} \sum_{q=n-1}^{N} f(n,q)Z^nY^q/n!$$

and

$$V = V(Z,Y) = \sum_{n=3}^{\infty} \sum_{q=n}^{N} v(n,q)Z^nY^q/n!$$

A result due to Gilbert [4] tells us that

$$R = e^F. \tag{1.1}$$

We use the "core and mantle" method due to Riddell [8] and Ford and Uhlenbeck [3] to find a functional equation satisfied by V. A simple account of this method is given in [6] pp.10,11, where it is used to find a functional equation satisfied by the e.g.f. of 2-connected labelled graphs. It has been modified by Walsh [9] to find a more complicated equation satisfied by the e.g.f. of 3-connected labelled graphs. Unlike the situation in these applications, however, the functional equation in the case of smooth graphs can be solved to find an explicit form for V. This is because the particular auxiliary e.g.f. introduced (that for rooted trees) has a simple inverse. I shall thus prove the following theorem.

Theorem 1: $V = \log R(Ze^{-ZY}, Y) - Z + \frac{1}{2}Z^2Y$.

Subsequently I give the various consequences including asymptotic expansions of or approximations to v(n,q)) which can be deduced from this or found otherwise. Where these can be found by methods already published in other applications, I give references to these methods rather than repetitive proofs.

2. Proof of Theorem 1

We write

$$G(X) = \sum_{n=1}^{\infty} n^{n-1}X^n/n! \qquad (2.1)$$

The e.g.f. for rooted labelled trees is then $g = G(ZY)/Y$. Again the inverse of $G(X)$ is

$$X = Ge^{-G}. \qquad (2.2)$$

This result is well known (see [7] or [11] for example); the simplest proof consists of defining G as the solution of (2.2) which vanishes with X and using Cauchy's theorem in an obvious way to prove (2.1). We remark that the series in (2.1) converges for $|X| < e^{-1}$, unlike the series for F and V which are formal. It follows that

$$Z = ge^{-gY}. \qquad (2.3)$$

Consider a connected (n,q) graph which is not a tree, so that $q \geq n$. We pluck the graph by removing each end-point and its adjacent line, continuing the process until we are left with a smooth graph. We can restore the original graph by rooting an appropriate tree (which may be the single point at the root) at each point on the smooth graph. It follows that

$$F(Z,Y) = W_{-1}(ZY)/Y + V(g,Y), \qquad (2.4)$$

where $W_{-1}(ZY)/Y$ is the e.g.f. for the number of (unrooted) labelled trees, so that

$$W_{-1}(X) = \sum_{n=1}^{\infty} n^{n-2}X^n/n!$$

Formula (8) of [11] gives us $W_{-1} = G - \frac{1}{2}G^2$, so that

$$W_{-1}(ZY) = gY - \tfrac{1}{2}g^2Y^2.$$

Hence, by (2.3) and (2.4), we have

$$F(ge^{-gY},Y) = g - \tfrac{1}{2}g^2Y + V(g,Y).$$

Since Z has now disappeared, we may replace g by Z and we have

$$V(Z,Y) = F(Ze^{-ZY},Y) - Z + \tfrac{1}{2}Z^2Y.$$

Theorem 1 follows by (1.1).

3. Partial differential equation satisfied by V.

We write

$$S_j = \sum_{n=j}^{\infty} Z^n e^{-nZY}(1+Y)^N/(n-j)!,$$

so that

$$S_0 = R(Ze^{-ZY},Y) = \exp(V + Z - \tfrac{1}{2}Z^2Y)$$

by Theorem 1. Differentiating partially with respect to Z (twice) and with respect to Y, we have

$$ZS_0(V_Z + 1 - ZY) = S_1(1-ZY),$$

$$Z^2S_0\{V_{ZZ} - Y + (V_Z+1-ZY)^2\} = S_1\{(1-ZY)^2 - 1\} + S_2(1-ZY)^2,$$

$$(V_Y - \tfrac{1}{2}Z^2)S_0 = -ZS_1 + \tfrac{1}{2}S_2(1+Y)^{-1}.$$

Eliminating S_0, S_1 and S_2, we have the following theorem.

Theorem 2. *The partial differential equation satisfied by V is*

$$2(1-ZY)^3(1+Y)V_Y =$$
$$= Z^2(1-ZY)(V_{ZZ}+V_Z^2) + Z^3Y^2(3-2ZY)V_Z + Z^3Y^2(1-ZY)^2. \qquad (3.1)$$

4. Direct combinatorial proof of Theorem 2

It is of some interest to give a direct combinatorial interpretation or proof of Theorem 2. To do so, we write (3.1) in the form

$$V_Y = Q_1 + Q_2 + Q_3 + Q_4 + Q_5, \tag{4.1}$$

where

$$Q_1 = \tfrac{1}{2}Z^3Y^2(1-ZY)^{-1}, \quad Q_2 = \tfrac{1}{2}ZV_Z(1-ZY)^{-1}\{(1-ZY)^{-2}-1-2ZY\},$$

$$Q_3 = \tfrac{1}{2}Z^2V_Z^2(1-ZY)^{-2}, \quad Q_4 = \tfrac{1}{2}Z^2V_{ZZ} - YV_Y,$$

$$Q_5 = \tfrac{1}{2}Z^2V_{ZZ}\{(1-ZY)^{-2} - 1\} .$$

We take the set $\mathcal{B}$ of all smooth (n,q+1) graphs in each of which one line is chosen as <u>special</u>. Since this choice can be made in q+1 ways in each graph, we have $|\mathcal{B}| = (q+1)v(n,q+1)$. We separate the set $\mathcal{B}$ into five mutually exclusive sub sets $\mathcal{B}_i$ $(1 \leq i \leq 5)$. The set $\mathcal{B}_1$ contains all the members of $\mathcal{B}$ in which no point is of degree greater than 2, i.e. every graph which consists of a single circuit. Each of the remaining members of $\mathcal{B}$ has at least one point of degree greater than 2. It follows that, in each of these graphs, the special line either belongs to a <u>suspended circuit</u> (i.e. a circuit all of whose points except one are of degree 2) or to a <u>suspended path</u> (i.e. a path of length one or more, all of whose internal points, if any, are of degree 2 and each of whose different end-points is of degree greater than 2). If the special line belongs to a suspended circuit, we put the graph in sub-set $\mathcal{B}_2$. If the special line belongs to a suspended path and if its removal disconnects the graph, we put the graph in set $\mathcal{B}_3$. In each of the remaining graphs, the removal of the special line leaves the graph connected; if the suspended path is of length one, i.e. consists of the special line alone, we put the graph in sub-set $\mathcal{B}_4$, if not, in sub-set $\mathcal{B}_5$,

Now consider the collection $\mathcal{C}$ of (n,q) graphs formed by removing the special line from each of the graphs in $\mathcal{B}$. (A collection, not a set, since in general some lines will occur more than once). The five sub-collections $\mathcal{C}_i$ $(1 \leq i \leq 5)$ are formed in the same way from the sub-sets $\mathcal{B}_i$. We have

$$|\mathcal{C}| = |\mathcal{B}| = (q+1)v(n,q+1)$$

and so the e.g.f. of $|\mathcal{C}|$ is V_Y.

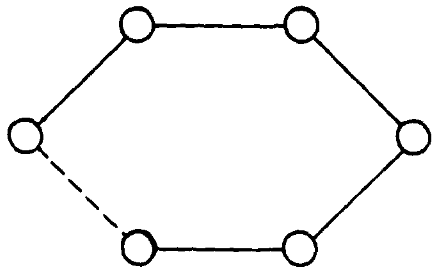
The figure shows a typical member of each of the $\mathscr{C}_i$. The shaded areas represent smooth sub-graphs and the broken line the "removed" line. The graphs in $\mathscr{C}_1$ are simple paths with n points ($n \geq 3$) and $q = n-1$ lines; hence $|\mathscr{C}_1| = \frac{1}{2}(n!)$ and the e.g.f. of $|\mathscr{C}_1|$ is

$$\frac{1}{2}\sum_{n\geq 3} Z^n Y^{n-1} = Z^3Y^2(1-ZY)^{-1} = Q_1.$$

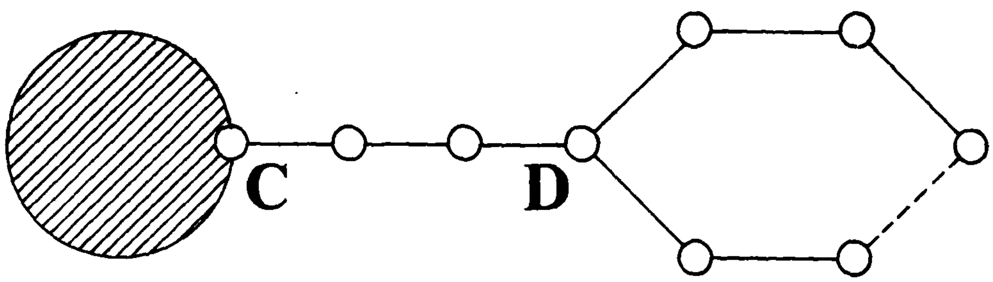
The graphs in $\mathscr{C}_4$ are connected and have all their points of degree not less than 2; i.e. they are smooth (n,q) graphs. Each such graph can be obtained by the removal of any one of N-q lines AB, each from an appropriate (n,q+1) graph; hence $|\mathscr{C}_4| = (N-q)v(n,q)$ and the e.g.f. of $|\mathscr{C}_4|$ is $\frac{1}{2}Z^2V_{ZZ} - YV_Y = Q_4$.

The graphs in $|\mathscr{C}_5|$ consist of a smooth graph with two of its points joined by a broken suspended path of original length greater than one. The e.g.f. of the number of smooth graphs with two distinguished points is $\frac{1}{2}Z^2V_{ZZ}$. The broken suspended path has its two end points unlabelled for counting purposes and one line missing; the number of such paths is $(n+1)!$, where $n \geq 1$, and the correspondine e.g.f. is

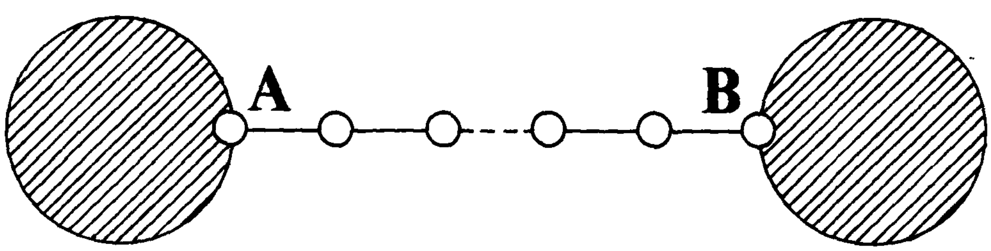
$$\sum_{n\geq 1}(n+1)Z^nY^n = (1-ZY)^{-2} - 1.$$



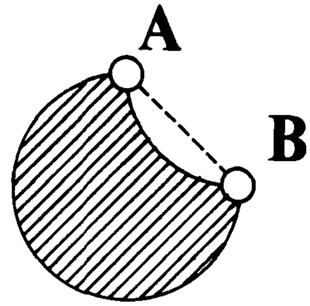
$\mathscr{C}_1$



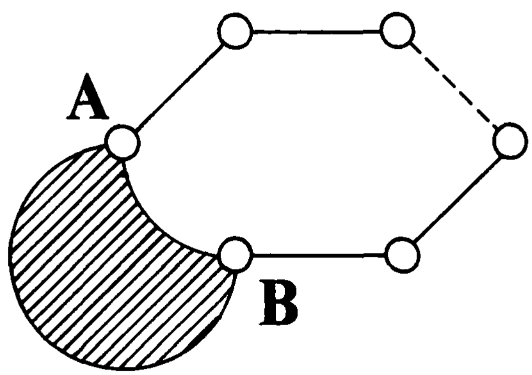
$\mathscr{C}_2$



$\mathscr{C}_3$



$\mathscr{C}_4$



$\mathscr{C}_5$

Hence the e.g.f. of $|\mathscr{C}_5|$ is Q_5, by the multiplication property of e.g.f.s (see [6]).

Each graph in $\mathscr{C}_3$ consists of two smooth sub-graphs each with a distinguished point (A,B in the figure) and a broken suspended path, of length 1 or more, each of whose end-points are, for counting purposes, unlabelled. The e.g.f. of the number of smooth graphs each with a distinguished point is ZV_Z and the e.g.f. of the broken suspended path is $\sum_{n\geq 0} (n+1)Z^nY^n = (1-ZY)^{-2}$. The e.g.f. of $|\mathscr{C}_3|$ is therefore Q_3, the $\frac{1}{2}$ occurring since otherwise each graph of $\mathscr{C}_3$ is counted twice.

Each graph in $\mathscr{C}_2$ consists of a smooth graph with a distinguished labelled point C, a suspended path CD (in which C is unlabelled, all other points are labelled and C and D may coincide) and a broken suspended circuit in which D is unlabelled. The corresponding e.g.f.s are ZV_Z, $(1-ZY)^{-1}$ and $\frac{1}{2}\sum_{n\geq 2} (n+1)Z^nY^n = \frac{1}{2}\{(1-ZY)^{-2} - 1 - 2ZY\}$ respectively. Hence the e.g.f. of $|\mathscr{C}_2|$ is Q_2. This completes the direct combinatorial proof of (3.1) in the form (4.1).

5. Sparsely-edged smooth graphs

If we put $X = ZY$ and

$$V_k = \sum_{n\geq 3} v(n,n+k)X^n/n!,$$

we have $V = \sum_{k\geq 0} V_k Y^k$. Substituing in (3.1) and equating coefficients of Y^{k+1}, we find that

$$\begin{aligned} 2(1-X)^3(XV'_{k+1} + (k+1)V_{k+1}) \\ = X^2(1-X)V''_k - (2-6X+3X^2)XV'_k - 2(1-X)^3 kV_k \\ + X^2(1-X)\sum_{h=0}^{k} V'_h V'_{k-h}, \end{aligned} \tag{5.1}$$

where dashes denote differentiation with respect to X.

Since $v(n,n) = \frac{1}{2}\{(n-1)!\}$, we have

$$V_0 = \tfrac{1}{2}\sum_{n\geq 3} X^n/n = -\tfrac{1}{2}\{\log(1-X) + X + \tfrac{1}{2}X^2\}.$$

With this, we can use the obvious integration of (5.1) to obtain a formula for V_{k+1} in terms of an integral involving V_h $(0\leq h\leq k)$, i.e. a recurrence formula satisfied by V_k for successive k. However in [12] I describe an alternative method of determining V_k as a finite sum of powers of $\phi = 1-X$, viz.

$$V_k = b_k\phi^{-3k} - c_k\phi^{-3k+1} + \sum_{s=-3k+2}^{2} c_{ks}\phi^s \quad (k\geq 1), \tag{5.2}$$

where the c_{ks} are those given in Theorem 4 of [11] and $b_k = c_{k,-3k}$, $c_k = -c_{k,1-3k}$. These can be calculated by computer by the methods described in [5] and [11] and, as shown in [13], $b_k = 3^k Z^{-k}(k-1)!\, d_k$, where d_k is the sequence described in Theorem 5 below.

6. Asymptotic approximations to v(n,q)

We suppose first that

$$\mu \equiv (q/n) - \tfrac{1}{2}\log n - \tfrac{1}{2}\log\log n \longrightarrow c \qquad (6.1)$$

as $n \to \infty$. Then Erdős and Rényi proved [2] that the proportion of (n,q) graphs in which the minimum degree is 2 tends to

$$D = 1 - \exp(-e^{-2c})$$

and that the proportion in which the minimum degree is 3 tends to 0. Again the same authors proved [1] that, if (6.1) is true, the proportion of (n,q) graphs which are connected tends to 1. The following theorem is immediate.

Theorem 3. *If (6.1) is true, then the proportion of (n,q) graphs which are smooth tends to D as $n \to \infty$, that is*

$$v(n,q)/B(N,q) \longrightarrow D.$$

Next let us suppose that $\mu \to +\infty$ as $n \to \infty$. From Theorem 1, we have

$$\exp(V+Z-\tfrac{1}{2}Z^2Y) = 1 + \sum_{n=1}^{\infty} Z^n e^{-nZY}(1+Y)^N/n!$$

We can then use the method of [10] to find an asymptotic expansion for v(n,q). The work is cumbrous in detail, but follows that of [10] in a fairly obvious manner. We write $N_r = \frac{1}{2}(n-r)(n-r-1)$. The result we obtain is as follows.

Theorem 4. *If* $\mu \to +\infty$ *as* $n \to \infty$, *then*

$$\begin{aligned} v(n,q) = {} & B(N,q) - n\{B(N_1,q)+(n-1)B(N_1,q-1)\} \\ & + B(n,2)\{B(N_2,q)+(2n-3)B(N_2,q-1)+(n-2)^2B(N_2,q-2)\} \\ & - B(n,3)\{B(N_3,q)+3(n-2)B(N_3,q-1)+3(n-2)(n-3)B(N_3,q-2) \\ & \qquad + ((n-3)^3+1)B(N_3,q-3)\} \\ & + O\{n^4B(N_4,q-4)\} \end{aligned}$$

With sufficient labour this expansion can be extended to any desired number of terms. The result can also be found by the use of the Inclusion-Exclusion theorem, but no more easily.

7. Asymptotic approximation to v(n,n+k).

We can also find an asymptotic approximation to $v(n,n+k)$ when $1 \leq k = o(n^{1/2})$. If we replace $\theta = 1-G$ in [13] by $\phi = 1-X$, the method is almost word-for-word identical with that used in [13] for $f(n,n+k)$. (Only the comparatively trivial Lemma 10 requires some alteration and §5 of [13] is not required.) We deduce the following theorem.

Theorem 5. *If* $1 \leq k = O(n^{1/2})$, *then*

$$v(n,n+k) = d_k (3/2)^k \{(k-1)!(n+3k-1)!/(3k-1)!\}\{1 + (k^2 n^{-1})\},$$

where

$$d_1 = d_2 = 5/36, \quad d_{k+1} = d_k + \sum_{h=1}^{k-1} d_h d_{k-h} / (k+1)B(k,h) \quad (k \geq 2). \tag{7.1}$$

In [13] we showed that d_k tends to a limit d as $k \to \infty$ and (by computing) that $d = 0.159155...$ Using Stirling's formula, we find a further theorem.

Theorem 6. *If* $k \to \infty$ *as* $n \to \infty$ *and* $k = o(n^{1/2})$, *then*

$$v(n,n+k) = a(18k^2)^{-k} e^{2k-n} n^{n+3k-\frac{1}{2}} \{1 + O(k^{-1}) + O(k^2 n^{-1})\},$$

where $a = d(6\pi)^{\frac{1}{2}} = 0.690986...$

We can also prove the following.

Theorem 7. *If* $k \to \infty$ *as* $n \to \infty$ *and* $k < (\text{[illegible]}n)^{\frac{1}{2}}(1-\varepsilon)$, *where* ε *is a positive number independent of* k *and* n, *then*

$$\log v(n,n+k) = (n+3k-\tfrac{1}{2})\log n + 2k-n - k\log(18k^2) + O(1).$$

Acknowledgment

The research reported here was supported by the European Research Office of the United States Army.

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Appendix 3

The k-connectedness of bipartite graphs

E.M.Wright

(To appear in J. London Math. Soc.)

Summary

We consider bipartite graphs on m red points and n blue points, where $m \leq n$, and prove that, for any fixed k, almost all such graphs (labelled or unlabelled) are k-connected as $n \to \infty$, provided $m > C \log n$, where C depends on k. If T_{mn} is the number of such unlabelled graphs, we show that $T_{mn} \sim 2^{mn}/(m!n!)$. If T'_{mn} is the number of such unlabelled graphs with the colours removed, then $T'_{mn} \sim T_{mn}$ if $m < n$ and $T'_{nn} \sim \frac{1}{2}T_{nn}$. We deduce that almost all bipartite graphs on p points in all, whether labelled or unlabelled, are k-connected and so prove a conjecture of Harary and Robinson.

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1. Introduction

In a recent paper [5] Harary and Robinson conjecture that almost all connected labelled bipartite graphs are 2-connected. Here I prove a number of results, of one of which the conjecture is an immediate corollary. Dr. Bollobás has very kindly shown me a paper of his [1] which also includes a theorem of which the same conjecture is an immediate corollary. The overlap between his paper and this one is however small.

Throughout we take k a fixed positive integer and write C for a suitable positive number (not always the same at each occurrence) which may depend on k, but not on p,m or n. The notation O( ) and o( ) refers to the passage of p or n to infinity (as the case may be) and the constants implied are of type C. We say that almost all graphs of a particular kind have a property if the proportion which have that property tends to 1 as $p \to \infty$ or as $n \to \infty$. Harary and Robinson's conjecture follows at once from the following theorem.

Theorem 1. *Almost all labelled (or unlabelled) bipartite graphs on p points are k-connected as $p \to \infty$.*

We first consider labelled (m,n) graphs, i.e. labelled bipartite graphs on m red points and n blue points, i.e. the red points are labelled $r_1, \ldots, r_m$

and the blue points $b_1, \ldots, b_n$. Each red point may be joined by just one line to each blue point or not so joined; no two points of the same colour are joined. The number of labelled (m,n) graphs is clearly $F_{mn} = 2^{mn}$. We take $m = m(n) \leq n$.

Theorem 2. _If_ $m > C \log n$, _almost all labelled_ (m,n) _graphs are_ k-_connected as_ $n \to \infty$.

Let T_{mn} be the number of unlabelled bipartite graphs on m red points and n blue points.

Theorem 3. _If_ $m > C \log n$, _then_

$$T_{mn} = \{2^{mn}/(m!n!)\}\{1 + o(1)\}.$$

Theorem 4. _If_ $m > C \log n$, _almost all bicoloured unlabelled_ (m,n) _graphs are_ k-_connected_.

Theorem 3 implies that, if $m > C \log n$, almost all bicoloured (m,n) graphs, labelled or unlabelled, have only the trivial automorphism. Theorem 4 follows from Theorems 2 and 3.

What is true if we remove the colours from the unlabelled graphs? Some of the resulting disconnected (m,n) graphs may then be isomorphic to one another but, in view of Theorem 4 (with k=1) there are almost none of these. If $m < n$, no two of the connected uncoloured graphs are isomorphic to one another, for a connected bipartite graph can only be bicoloured in one way (apart from an interchange of the two colours, impossible here since the m points are originally coloured red). This argument fails, of course, if $m = n$. Let T'_{mn} be the number of non-isomorphic uncoloured unlabelled (m,n) graphs. We have proved the first part of the following theorem.

<u>Theorem 5</u>. <u>If</u> $m > C \log n$, <u>then</u>

$$T'_{mn} \sim 2^{mn}/(m!n!) \qquad (m < n); \qquad T'_{nn} \sim 2^{n^2-1}/(n!)^2.$$

The second part is proved in § 4.

There is a sense in which our theorems so far are "weak", since they refer to almost all (m,n) graphs, make no reference to the number of lines and do not find the "threshold" for connectedness in the sense of [3] or the asymptotic numbers of unlabelled (m,n;E) graphs (E being the number of lines) as in [6,7] for ordinary graphs. So far as simple connectedness of labelled bipartite graphs is concerned, we have the following theorem (which I do not prove here).

Theorem 6. Let $m \leq n$,

$$E = n\{m - (m-\gamma)e^{-(\log n)/m}\} \tag{1.1}$$

and $m \to \infty$ and $\gamma \to c$ as $n \to \infty$. If $\{(n-m)\log n\}/n \to b$, then the proportion of labelled $(m,n:E)$ graphs which are connected tends to $\exp\{-e^{-c}(1+e^{-b})\}$. If $\{(n-m)\log n\}/n \to \infty$, then this proportion tends to $\exp(-e^{-c})$.

Professors Klee and Larman and I hope to publish the proof of Theorem 6 in a joint paper. This proof is distinctly more complicated than that of Theorem 2 of the present paper. It is interesting to note that, if $m/\log n$ is bounded above as $n \to \infty$, then E in (1.1) is not $o(mn)$, whereas the condition that $E = o(n^2)$ plays an essential role in [2].

Let $\alpha = \alpha(m,n)$ be the proportion of labelled (m,n) graphs which are connected. We can deduce from Theorem 6 or prove directly that, if $n2^{-m} \to w$ as $m,n \to \infty$ with $m \leq n$, then $\alpha(m,n) \to e^{-w}$ and correspondingly, if $n2^{-m} \to \infty$, then $\alpha \to 0$. (Dr. Bollobás tells me that he knew this and that he thinks that he has seen a published proof.) The result shows that the condition $m > C \log n$ in Theorem 2, while not best possible, is nearly so.

There seems to be no serious obstacle to extending the methods of [4] on k-connectedness to labelled bipartite $(m,n;E)$ graphs, though the extra variable m certainly introduces complication. The extension of [6,7] (and especially [7], the methods of which are themselves unattractively complicated) to the bipartite case may be more troublesome and so far I have only partial results. Thus the point of the present paper is that, while the theorems are probably not best possible, the proofs are relatively simple and straightforward.

In what follows we write $B(h,k) = h!/\{k!(h-k)!\}$.

2. Proof of Theorem 2

If an (m,n) graph is not k-connected, there are $k-1$ points, say r red and s blue, where $r+s = k-1$, such that, if these points and all lines adjacent to them are removed, we are left with an (x,y) graph and an $(m-r-x, n-s-y)$ graph, unconnected to one another. There are therefore $\Lambda = x(n-s-y) + y(m-r-x)$ lines which cannot occur in the original graph. The number of such original labelled graphs is therefore at most

$$B(m,r)B(n,s)\sum{}' B(M,x)B(N,y)2^{mn-\Lambda}, \qquad (2.1)$$

where $\sum{}'$ denotes summation over all x,y such that $0 \leq x \leq M = m-r$, $0 \leq y \leq N = n-s$, $1 \leq x+y \leq M+N-1$. We can clearly choose $x \leq \frac{1}{2}M$, but not then choose $y \leq \frac{1}{2}N$. However, if $y > \frac{1}{2}N$, write $y' = N-y < \frac{1}{2}N$; we have

$$\Lambda(x,y) = x(N-y) + y(M-x) = xN + y(M-2x)$$
$$\geq xN + y'(M-2x) = \Lambda(x,y').$$

Hence the number (2.1) is at most

$$2B(m,r)B(n,s)\sum{}'' B(M,x)B(N,y)2^{mn-\Lambda(x,y)},$$

where $\sum{}''$ denotes summation over all x,y such that $0 \leq x \leq \frac{1}{2}M$, $0 \leq y \leq \frac{1}{2}N$, $1 \leq x+y$. The proportion of such graphs among all labelled (m,n) graphs is therefore at most

$$\Omega_{r,s} = 2B(m,r)B(n,s)\sum{}'' B(M,x)B(N,y)2^{-\Lambda(x,y)} \leq C\sum{}'' e^{-\Delta(x,y)}/(x!y!),$$

where

$$\Delta(x,y) = \tfrac{1}{2}(Nx+My)\log 2 - (r+x)\log m - (s+y)\log n$$
$$\geq C(m - C\log n)(x+y),$$

provided $m > C \log n$. Hence $\Omega_{rs} = o(1)$ and Theorem 2 follows.

3. Proof of Theorem 3: the unlabelled case

By the so-called (see [7]) Burnside Lemma we have

$$m!n!T_{mn} = F_{mn} + \sum_{\rho,\sigma} F(\rho,\sigma), \qquad (3.1)$$

where $F(\rho,\sigma)$ is the number of labelled (m,n) graphs invariant under the permutation ρ of the labels $r_1,\ldots,r_m$ and the permutation σ of the labels $b_1,\ldots,b_n$ and $\sum_{\rho,\sigma}$ denotes summation over all ρ and all σ, except the pair in which $\rho = \sigma = I$, the identity.

The permutation ρ can be expressed uniquely as a product of disjoint cycles, of which p_j are of length j; similarly σ is a product of disjoint cycles, of which q_j are of length j. We have

$$m = \sum_j jp_j, \qquad n = \sum_j jq_j.$$

Let π be the corresponding permutation of the lines joining the $\{r_i\}$ to the $\{b_i\}$ and P_j the number of cycles in π of length j. Then

$$mn = \sum_j jP_j.$$

Consider those ρ in which just a of the r_i are changed, so that $p_1 = m-a$. There are at most $B(m,a)a! \leq m^a$ such ρ. Similarly there are at most n^b permutations σ in which $q_1 = n-b$. For such ρ and σ, we have

$$P_1 = (m-a)(n-b).$$

The number of labelled (m,n) graphs invariant under any such pair ρ, σ is $2^{\sum P_j}$, since the j lines affected by any j-cycle of π are either all present or all absent. Now

$$\sum P_j = P_1 + \sum_{j\geq 2} P_j \leq P_1 + \tfrac{1}{2}\sum_{j\geq 2} jP_j = \tfrac{1}{2}(P_1 + mn)$$
$$= mn - \tfrac{1}{2}an - \tfrac{1}{2}bm + \tfrac{1}{2}ab.$$

From (3.1), we have

$$m!n!T_{mn} = 2^{mn} + J, \qquad J = \sum_{\rho,\sigma} 2^{\sum P_j}. \qquad (3.2)$$

We remark that $a = 0$ or $a \geq 2$ and $b = 0$ or $b \geq 2$. Then

$$J2^{-mn} \leq \sum_{a\geq 2} m^a 2^{-\frac{1}{2}an} + \sum_{b\geq 2} n^b 2^{-\frac{1}{2}bm} + \sum_{a\geq 2, b\geq 2} m^a n^b 2^{-\frac{1}{2}(an+bm-ab)}$$

Now, if $2 \leq a \leq m$ and $2 \leq b \leq n$,

$$an + bm - ab \geq \tfrac{1}{2}(an + bm)$$

and so Theorem 3 follows if we can show that

$$\sum_{a\geq 2} m^a 2^{-\frac{1}{4}an} = o(1), \qquad \sum_{b\geq 2} n^b 2^{-\frac{1}{4}bm} = o(1).$$

These are clearly true if

$$m \log 2 - 4 \log n \to \infty \qquad (3.3)$$

as $n \to \infty$, i.e. if $m > C \log n$ for an appropriate C.

4. Proof of the second part of Theorem 5

As before, we may disregard the disconnected graphs. Consider a connected bicoloured (un)labelled (n,n) graph G_1. Recolour the red points blue and the original blue points red and so obtain a graph G_2. If G_1 and G_2 are not isomorphic to one another (i.e. isomorphic red to red and blue to blue), we say that they are a reciprocal pair. If they <u>are</u> isomorphic to one another in this sense, we say that G_1 is self-reciprocal. If then we remove the colours from every connected unlabelled (n,n) graph, we have to discard one of every pair of reciprocal graphs to obtain the connected members of the collection enumerated by T'_{nn}. The following lemma suffices to prove that $T'_{nn} = \frac{1}{2}T_{nn}\{1 + o(1)\}$, from which the second part of Theorem 5 follows at once.

<u>Lemma</u>. <u>If</u> S <u>is the set of self-reciprocal bicoloured labelled</u> (n,n) <u>graphs, then</u> $|S| = o(2^{n^2})$.

Consider the self-reciprocal bicoloured labelled (n,n) graph G. It has an automorphism M (red to blue, blue to red) which maps $r_1,\ldots,r_n$ on to $\tau_1(b_1,\ldots,b_n)$, where τ_1 is a permutation of the suffixes $1,2,\ldots,n$, and $b_1,\ldots,b_n$ on to $\tau_2(r_1,\ldots,r_n)$. Hence $\tau_1(b_1,\ldots,b_n)$ is mapped on to $\tau_2\tau_1(r_1,\ldots,r_n)$. Two cases arise. If $\tau_2\tau_1$ is not the identity, we repeat the mapping M.

Then, under the mapping M^2, $r_1,\ldots,r_n$ maps on to $\tau_2\tau_1(r_1,\ldots,r_n)$ and $b_1,\ldots,b_n$ on to $\tau_1\tau_2(b_1,\ldots,b_n)$, Hence the graph G is invariant under the permutation $\rho = \tau_2\tau_1$ of the suffixes of the r_i and $\sigma = \tau_1\tau_2$ of the suffixes of the b_i. Thus G is invariant under the non-identity permutation ρ of the labels of the red points and the permutation σ (also non-identity) of the labels of the blue points. It is therefore one of the bicoloured labelled (n,n) graphs counted (perhaps more than once) in the sum J of (3.2), which is $o(2^{n^2})$.

Now consider those G for which $\tau_2\tau_1 = I$, These have a red on blue, blue on red automorphism in which r_i maps onto $b_{\tau_1(i)}$ and $b_{\tau_1(i)}$ on to r_i. There are n! possible choices of τ_1. Hence there are at most n!D graphs G of this type, where D is the number of graphs G in which r_i maps onto b_i and b_i onto r_i for every i. In such a graph, r_i is (or is not) joined to b_i and the line r_ib_j is present if and only if the line b_ir_j is present. Hence there is a (1,1) correspondence between these G and the graphs (not necessarily bipartite) on n points with a possible loop at each point. It follows that $D = 2^{\frac{1}{2}n(n+1)}$, so that $n!D = o(2^{n})$. The lemma is immediate.

5. Proof of Theorem 1

Let F_p be the number of bipartite uncoloured graphs on p labelled points and, as before, $F_{mn} = 2^{mn}$ the number of bipartite graphs on m red points labelled $r_1, \ldots, r_m$ and n blue points labelled $b_1, \ldots, b_n$. We remove the restriction that $m \leq n$. Clearly

$$F_p = \tfrac{1}{2} \sum_{m=1}^{p-1} B(p,m) F_{m,p-m}.$$

If f_p and f_{mn} are the corresponding numbers of labelled k-connected graphs, we have

$$f_{mn} = F_{mn}\{1 + o(1)\}$$

by Theorem 2, provided min $(m,n) > C \log \max (m,n)$. But

$$\sum_{m \leq C \log p} B(p,m) F_{m,p-m} = \sum_{m \leq C \log p} B(p,m) 2^{m(p-m)}$$
$$\leq C 2^{p(1+C \log p)} = o(F_{[\frac{1}{2}p], p-[\frac{1}{2}p]}) = o(F_p)$$

and so we have

$$f_p = \tfrac{1}{2} \sum_{m=1}^{p-1} B(p,m) f_{m,p-m} = F_p \{1 + o(1)\}.$$

This is the labelled case of Theorem 1.

If T_p is the number of unlabelled bicoloured bipartite graphs on p points, we have

$$T_p = \sum_{m=1}^{p-1} T_{m,p-m}$$

and we can prove that almost all the T_p graphs are k-connected as above. Similarly, with minor variations, for unlabelled uncoloured bipartite graphs.

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